Coulomb Interactions in a Strong Magnetic Field. II. Nonadiabatic Case. Line Profile of Cyclotron Radiation*

ROBERT GOLDMAN
Physics Department, Yale University, New Haven, Connecticut†
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Using the generalized quantum treatment of pressure broadening by Anderson, we derive an expression including the effects of both adiabatic and nonadiabatic transitions for the emission spectrum of cyclotron radiation from free-charged particles undergoing Coulomb interactions with fixed scatterers in typical plasma situations. After adapting the formalism to account for the continuous nature of the energy states parallel to the magnetic field, and after including the energy dependence of the dipole matrix element, one obtains half-width values for the cyclotron line in the case of a strong magnetic field. Extension to the weak field case can then be made in a plausible manner, resulting in the ordinary Coulomb cross section for vanishing magnetic field strength.

1. INTRODUCTION

PREVIOUSLY[1] the variation in the spatial part of the wave function in the determination of the cyclotron radiation spectrum was neglected. In the limiting case where the kinetic energy E of the particle is large compared with the cyclotron photon energy, the major portion of the cyclotron line's half-width is caused by this variation. It is the purpose of this paper to derive explicitly the contribution from such transitions to the scattering cross section, and to calculate in particular the half-width of the cyclotron line. Bremsstrahlung contributions, which do not significantly alter the frequency dependence of the spectrum in the neighborhood of the cyclotron resonance will be considered in a subsequent paper. It is assumed that the scattering centers are stationary so that, for instance, the contributions from electron-electron collisions are omitted.

In this paper and in I the variations resulting from the Coulomb interaction in the time-dependent parts of the phase factor of the magnetic field eigenfunctions are called adiabatic, whereas the changes in the various spatial quantum states are termed nonadiabatic. For the weak collisions which dominate in the study of coulomb scattering, those variations which we have termed adiabatic and which are calculated using the unperturbed eigenfunctions for the electron in the magnetic field, agree closely with the results from the ordinary adiabatic treatment. The latter results use eigenfunctions of the instantaneous particle Hamiltonian during the scattering, with the assumption that there are no transitons between different eigenfunctions. For weak collisions the effects which we have called adiabatic and nonadiabatic are additive.

To account for the nonadiabatic contributions to cyclotron radiation it is necessary to calculate the transition probability as a result of coulomb interactions between quantum states for the electron in a strong magnetic field. A similar approach was used by Tannenwald[2] who was able to obtain approximate values for transition probabilities. However, the similarities and differences between the situation in a strong magnetic field and the case of ordinary coulomb scattering were obscured due to the inaccuracies in the values for the transition probabilities. More recently, the collision integral has been studied in connection with particle diffusion across a magnetic field[3] where it was found necessary to introduce a long-distance cutoff as a result of shielding interactions in the plasma.

Treating the interaction in the Born approximation we calculate the transition probabilities to high accuracy, and obtain a clear picture of maximum impact parameters and deviations from coulomb scattering. Results are then extended to lower energies where the Born approximation does not hold.[4]

We begin by summarizing Anderson's line-broadening theory[5] altering the formalism as we go along to account for the continuous nature of the electron energies parallel to the magnetic field (Secs. 2 and 3). In Sec. 4 we justify the use of semiclassical approximations by comparison with the strictly quantum-mechanical approach. After evaluating the general form of the cross section in Sec. 5, we specialize to the case of a strong magnetic field (Secs. 6 and 7). After giving a physical interpretation in Sec. 8 we turn to the case of weak magnetic field in Sec. 9. Finally in Sec. 10 the results are summarized for ready reference, and in Sec. 11 illustrative examples are presented.

2. FORMAL SOLUTION OF THE PROBLEM

Classically one has for the intensity of emission $I(\omega)$ at frequency ω for the time interval from 0

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† Present address: Princeton University Plasma Physics Laboratory, Princeton, New Jersey.
[1] R. Goldman and L. Oster, Phys. Rev. **129**, 1469 (1963). Hereafter this reference will be called Paper I.

[2] L. M. Tannenwald, Phys. Rev. **113**, 1396 (1959).
[3] V. M. Eleonskii, P. S. Zyryanov, and V. P. Silin, Zh. Eksperim. i Teor. Fiz. **42**, 896 (1962) [English translation: Soviet Phys.—JETP **15**, 619 (1962)].
[4] E. J. Williams, Rev. Mod. Phys. **17**, 217 (1945).
[5] P. W. Anderson, Phys. Rev. **76**, 647 (1949).

to T

$$I(\omega)=\text{const}\,\omega^4\left|\int_0^T \mathbf{\mu}(t)\times\exp(-i\omega t)dt\int_0^T \mathbf{\mu}(t)\exp(-i\omega t)dt\right|, \quad (1a)$$

where $\mathbf{\mu}$ is the classical dipole moment of the radiating particle.

On the other hand, the theory of Anderson[5] gives as a quantum-mechanical expression for the time average of spontaneous emission

$$I(\omega)=\text{const}\lim_{T\to\infty} T^{11}\sum_\alpha \text{Tr}\left\{\rho_0\int_0^T dt\int_0^T dt' \times\exp[-i\omega(t'-t)]\mu_\alpha(t)U^{-1}(t\to t') \times\mu_\alpha(t)U(t\to t')\right\}. \quad (1b)$$

Here ρ_0 is the density matrix, μ_α is the spatial component of the dipole matrix element in an arbitrary α direction, and U is a time-development matrix which will be defined shortly.

Taking the Hamiltonian for a beam of electrons in a magnetic field to be

$$H=H_0+H', \quad (2)$$

where H_0 is the magnetic contribution and H' is the collision interaction contribution, the matrices U and U_0 are defined such that

$$\phi_{n'}(t)=U_0(t)\phi_n(0), \quad (3)$$

$$\phi_{n''}(t)=U(t)\phi_n(0), \quad (4)$$

where $\phi_n(t)$ is the wave function at time t due to the interaction of the magnetic field alone, and $\phi_{n''}(t)$ is the wave function due to both the magnetic field and the coulomb collision interactions. (Here H' may be thought of as explicitly time-dependent; we agree however to choose the time dependence of H' so that the transition probabilities for single electron-ion interactions are identical to those of the Born approximation. The difference between this variation and that from the straight-line approximation will be shown in Sec. IV.)

For a quantum state a with energy E_a we have

$$[a|U_0(t)|b]=\delta_{ab}\exp[E_a t/i\hbar]. \quad (5)$$

Defining a matrix operator T by

$$T=U_0^{-1}U \quad (6)$$

we obtain instead of Eq. (1b)

$$I(\omega)=\text{const}\lim_{T\to\infty}\frac{1}{T}\sum_\alpha \text{Tr}\left\{\rho_0\int_0^T dt\int_0^T dt' \times\exp[-i\omega(t'-t)]\mu_\alpha(t)T^{-1}U_0^{-1}\mu_\alpha U_0 T\right\}. \quad (7)$$

Taking the average of the trace for a given quantum state of the electron over all possible ionic collision locations, one has on performing the first time integration

$$I(\omega)=\text{const}2\,\text{Re}\sum_\alpha \text{Tr}\left[\rho_0\int_0^\infty d\tau\exp(-i\omega\tau) \times\langle\mu_\alpha T^{-1}U_0^{-1}\mu_\alpha U_0 T\rangle_{\text{av}}\right], \quad (8)$$

with Re denoting the real part of the expression to follow, and $\tau=|t'-t|$. Removing U_0 with the aid of Eq. (5), we have

$$I(\omega)=\text{const}2\,\text{Re}\int_0^\infty \sum_{a,b,c,d}\sum_\alpha \exp[(\omega_{cd}-\omega)i\tau]\rho_{aa} \times\{(a|\mu_\alpha|b)[b|T^{-1}(\tau)|c](c|\mu_\alpha|d)\cdot[d|T(\tau)|a]\}_{\text{av}}d\tau, \quad (9)$$

where the sum over a, b, c, and d corresponds to the trace operation, and the frequency

$$\omega_{cd}=(E_c-E_d)/\hbar \quad (10)$$

refers to the quantum states c and d.

The problem is therefore to evaluate for a given a and b (with E_c-E_d fixed)

$$[b|T^{-1}(\tau)|c](c|\mu_\alpha|d)[d|T(\tau)|a]_{\text{av}}=f_{b,a}(\tau). \quad (11)$$

It can be shown rigorously to an accuracy of[6] $(E_b-E_c)/(E_b+E_c)$ that

$$[b|T^{-1}(\tau)|c] =\left\{b\left|\exp\left[(-i\hbar)^{-1}\int_0^\tau U_0^{-1}H'U_0 dt\right]\right|c\right\} \quad (12)$$

(t is now a dummy variable).

The collision Hamiltonian for a given electron is made up of the sum of the separate Hamiltonians for each ion with the given electron. Therefore we have

$$[b|T^{-1}(\tau)|c] =\left\{b\left|\prod_{\nu=1}^N\exp\left[-(i\hbar)^{-1}\int_0^\tau U_0^{-1}H_\nu U_0 dt\right]\right|c\right\}, \quad (13)$$

where the running index designates the νth ion in the plasma. For the νth ion, one obtains

$$\left\{b\left|\exp\left[-(i\hbar)^{-1}\int_0^\tau U_0^{-1}H_\nu U_0 dt\right]\right|c\right\} \times\left[b\left|1-(i\hbar)^{-1}\int_0^\tau U_0^{-1}H_\nu U_0 dt+(2!)^{-1}(i\hbar)^{-2} \times\int_0^\tau U_0^{-1}H_\nu U_0 dt\int_0^\tau U_0^{-1}H_\nu U_0 dt\right|c\right], \quad (14)$$

[6] The error stems from the neglect of noncommuting terms in H_0+H_1.

since H_ν is equal to zero except for some time centered around an arbitrary τ.

Letting $\tau=t_0+t'$, we may write

$$[a|U_0(\tau)|b]=\delta_{ab}\exp[(i\hbar)^{-1}E_a(t_0+t')]. \quad (15)$$

Then the above expression (14) is of the form[7]

$$\exp[(i\hbar)^{-1}(E_c-E_b)t_0]K_1(b,c), \quad (16)$$

where K_1 depends on the quantum states b and c and the Hamiltonians, but not on t_0. For the same reasons the corresponding element for $T(\tau)$ is of the form[7]

$$\exp[(i\hbar)^{-1}(E_d-E_a)t_0]K_1(d,a). \quad (17)$$

Since the collision with ion ν can occur randomly in time, the average of

$$(b|T^{-1}|c)(d|T|a) \quad (18)$$

over t_0 tends to zero unless

$$(E_c-E_b)-(E_d-E_a)=0. \quad (19)$$

Since this is true for each collision, terms which do not satisfy this condition, i.e., terms for which

$$(E_c-E_b)-(E_d-E_a)=n\hbar\omega_c, \quad n>0, \quad \text{integer}, \quad (20)$$

vanish for all time intervals larger than the order of $2\pi/\omega_c$. Taking

$$T(\tau+\delta)=T(\tau)T(\delta), \quad (21)$$

one finds from Eq. (11) that

$$f_{b,a}(\tau+\delta)=\sum_{c,e}[b|T^{-1}(\tau)|e][c|T^{-1}(\delta)|e] \times(e|\mu|f)[f|T(\delta)|d][d|T(\tau)|a]. \quad (22)$$

In physical terms, Eq. (21) expresses the fact that interactions occurring in the time interval are independent of previous interactions. Hence

$$f_{b,a}(\tau+\delta)-f_{b,a}(\tau)=\sum_{c,e}(b|T^{-1}|c)(c|\mu|d)(d|T|a) \times\{(c|T^{-1}|e)(e|\mu|f)(f|T|d)/(c|\mu|d)-1\}. \quad (23)$$

Since the total particle angular momentum is not a constant of the motion for the case of a charged particle in a strong external magnetic field, the analysis must now depart from previous spectral line treatments. Letting the expression within the brackets equal A_e, i.e.,

$$(c|T^{-1}|e)\{(e|\mu|f)/(c|\mu|d)\}(f|T|d)-1, \quad (24)$$

and assuming that the difference between states b and c has a negligible effect on A_e, we will show later that for short times δ,

$$A=\sum_e A_e=\gamma\delta, \quad (25)$$

where γ is a constant. Therefore we obtain

$$[df_{b,a}(\tau)/d\tau]\delta=\gamma f_{b,a}(\tau), \quad (26)$$

for which

$$f_{b,a}(\tau)=\beta\exp(\gamma\tau), \quad (27)$$

with the value of the constant β gotten from the condition that

$$f_{b,a}(0)=(b|\mu|a), \quad (28)$$

so that

$$f_{b,a}(\tau)=(b|\mu|a)\exp(\gamma\tau). \quad (29a)$$

Alternately in terms of a cross section σ, we have

$$f_{b,a}(\tau)=(b|\mu|a)\exp(-Nv_z\sigma\tau), \quad (29b)$$

where σ is defined by

$$\sigma=-\gamma(Nv_z)^{-1} \quad (30)$$

with v_z the particle velocity parallel to the magnetic field, and N the fixed scatterer density.

The problem of calculating the spontaneous emission probability [cf. Eqs. (9), (11), (29)] is therefore reduced to the calculation of γ or σ. It should be noted that the operator $T(\delta)$ involves the effect of interactions from randomly distributed ions on an electron wave function whose guiding center location is determined in space with respect to an arbitrary origin. The average of this effect is taken to be the same as the average gotten from randomly distributed electrons which interact with ions located at the origin of an arbitrary reference frame in space.

3. SPECIALIZATION OF THE CALCULATION TO THE CASE OF ELECTRONS IN A STRONG MAGNETIC FIELD

Until now the transition probabilities were defined per single final quantum state. For our applications however, the transition probability per final state must be multiplied by the number of final quantum states consistent with the energy uncertainty of the initial state.

The electron eigenstates are characterized by a quantized energy E_b perpendicular to the magnetic field and a continuous energy E_z parallel to the magnetic field. In addition there is a quantum number s which accounts for the randomness of location of the electron guiding centers with respect to a fixed ion at the coordinate origin.[8]

Starting with a single quantum state with energy uncertainty dE_z at energy $E_z=p_z^2/(2m)$, we have

$$dp_z\propto dE_z/p_z.$$

Since the number of quantum states per unit volume is proportional to dp_z, a single quantum state at E_z with an energy uncertainty dE_z corresponds to (p_z/p_z') quantum states at $E_z'=p_{z'}^2/2_m$ with the same energy uncertainty. Hence in Eq. (23) the transition probability involving states e and c or f and d must be

[7] C. J. Tsao and B. Curnutte, J. Quant. Spectry. & Radiative Transfer 2, 41 (1961).

[8] See Appendix A.

weighted by a state density factor of

$$(v_c/v_e)\approx(v_d/v_f)\,, \tag{31}$$

where v_m is the velocity component of state m parallel to the magnetic field.

Letting ϵ_e designate the energy of state e perpendicular to the magnetic field, we have

$$\epsilon_e=\epsilon_c+m\hbar\omega_c\,,\quad m \text{ integer}. \tag{32}$$

Designating

$$T_{m,c,s}=[c,s|T^{-1}(\delta)|e][f|T(\delta)|d](v_c/v_e)\,, \tag{33}$$

where s is a degenerate quantum number used to locate the electron guiding center[8] and

$$\mu_m=(e|\mu|f)/(c|\mu|d)\,, \tag{34}$$

we may write the quantity A defined by Eq. (25) as

$$A=\sum_{m,s}(T_{m,c,s}\mu_m-1)P(s)\,, \tag{35}$$

where $P(s)$ is the probability for the electron to be in a state characterized by s with respect to a given ion.

Expanding μ_m, i.e.,

$$\mu_m=1+(\partial\mu_m/\partial m)|_{m=0}m+(2)^{-1}(\partial^2\mu_m/\partial m^2)|_{m=0}m^2, \tag{36}$$

one has for Eq. (35)

$$A=\sum_{m=m_1}^{m=m_2}\sum_s(T_{m,c,s}-1)P(s)+\sum_{m=m_1}^{m=m_2}\sum_s T_{m,c,s}\frac{\partial\mu_m}{\partial m}\bigg|_{m=0}mP(s)+\sum_{m=m_1}^{m=m_2}\sum_s T_{m,c,s}\frac{1}{2}\frac{\partial^2\mu_m}{\partial m^2}\bigg|_{m=0}m^2P(s). \tag{37}$$

The running index m may have positive or negative values; cf. Eq. (32). Positive values of m correspond to a scattering process in which the energy perpendicular to the magnetic field is increased; negative values to the opposite situation. It will be shown later that the upper and lower limits, m_1 and m_2, are determined by the effective maximum scattering angle, $\pi/2$, for the Born approximation to coulomb scattering.

The second sum in Eq. (37) may be simplified to read

$$\sum_{m=0}^{m_2}\sum_s\frac{\partial T}{\partial m}m,c,s\bigg|_{m=0}\frac{\partial\mu_m}{\partial m}\bigg|_{m=0}2m^2P(s) \tag{38}$$

if $m_1=m_2$. The condition on the limits of m will be verified later. The differential $\partial/\partial m$ operating on $T_{m,c,s}$ does not act on the order of the function $T_{m,c,s}$.

For the first and third sums it is sufficient to use the value of $T_{m,c,s}$ obtained from the semiclassical straight-line approximation, since in these cases there are no terms with opposite sign involving the same initial state but different final states. However, for the second summation we must use a more exact, purely quantum mechanical expression. In the following section the semiclassical and quantum-mechanical approaches to scattering problem will be obtained and compared.

4. COMPARISON OF SEMICLASSICAL AND QUANTUM TREATMENTS

In the *semiclassical approximation* we view each electron as a macroatom, quantized perpendicular to the magnetic field and representable by a probability density of the form $\delta(z-vt)$ parallel to the magnetic field, where z and v are, respectively, the position and velocity parallel to the field and t is the time. Then one may calculate the transition probability between two atomic states with the same velocity along the field but with quantum states (n,s) and $[n',\,s+(n'-n)]$.

For a single ion-electron encounter the probability amplitude becomes[9] to first order[10]

$$C_{n,k,s\to n',k',s'}(s>n)=(2e^2/i\hbar v)K_{n'-n}[qs^{1/2}]I_{n'-n}[qn^{1/2}]\,, \tag{39a}$$

$$C_{n,k,s\to n',k',s'}(s<n)=(2e^2/i\hbar v)K_{n'-n}[qn^{1/2}]I_{n'-n}[qs^{1/2}]. \tag{39b}$$

The symbol q is defined by

$$q=(E_b/E_z)^{1/2}(n'-n)/n^{1/2}\text{ with }E_b=n\hbar\omega_c\,,\ n<n'. \tag{40}$$

The transition probability in terms of $C_{n,k,s\to n',k',s'}$ is simply $|C_{n,k,s\to n',k',s'}|^2$.

In the *pure quantum treatment*, on the other hand, we consider the wave function for a beam of electrons with quantum numbers (n,k,s) and calculate the transition probability to a beam with quantum numbers (n',k',s') where

$$\hbar^2k^2/(2m)+n\hbar\omega_c=\hbar^2k'^2/(2m)+n'\hbar\omega_c. \tag{41}$$

Equation (41) states the conservation of energy. We note that energy changes *parallel to the magnetic field* are now taken into account.

From Fermi's second golden rule[11] we have for w, the transition probability per unit time for a beam of particles with unit density

$$w=(2\pi/\hbar)|\langle n',k',s'|e^2(r^2+z^2)^{-1/2}|n,k,s\rangle|^2dn/dE\,, \tag{42}$$

where dn is the state number per energy interval dE. Taking a volume with unit length in the direction parallel to the magnetic field, we have for the number of modes necessary for completeness

$$2\pi n=k\,,\quad n \text{ integer}. \tag{43}$$

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[9] R. Goldman and L. Oster, Aeronautical Res. Labs. of the Office of Aerospace Res., U. S. Air Force, Wright–Patterson AFB, Ohio, 1963 (to be published).

[10] L. M. Tannenwald, Phys. Rev. **113**, 1396 (1959), has carried out a similar calculation in the limit $s=n$. His Eq. (17) agrees with out results except for a factor of two which is hardly of any importance.

[11] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950), p. 142.

Since $k=(p/\hbar)$, it follows immediately that

$$dn=dp/2\pi\hbar. \quad (44)$$

Defining

$$v_f=\hbar k'/m\,,\quad v_i=\hbar k/m\,, \quad (45)$$

we obtain for the transition probability (w/v_i) per incident particle

$$w/v_i=|\alpha(n,k,s,n'-n)|^2/(v_iv_f)\,, \quad (46)$$

where

$$\alpha(n,k,s,n'-n)=\langle n',k',s'|e^2/(\hbar(r^2+z^2)^{1/2}|n,k,s\rangle. \quad (47)$$

Here r and z are radial and axial coordinates of the cylindrical reference frame centered at the ion location, with the z axis parallel to the magnetic field.

Equation (46) is the same as

$$w/v_i=(2e^2/\hbar)^2K_{n'-n}{}^2[qs^{1/2}]I_{n'-n}{}^2[qn^{1/2}]v_i^{-1}v_f^{-1}\,,\quad s>n\,, \quad (48a)$$

$$=(2e^2/\hbar)^2K_{n'-n}{}^2[qn^{1/2}]I_{n'-n}{}^2[qs^{1/2}]v_i^{-1}v_f^{-1}\,,\quad s<n\,, \quad (48b)$$

where I and K are the usual Bessel functions. The value E_b is that used in Eq. (40); $E_z\approx(m/4)(v_i^2+v_f^2)$. Since Eq. (48) is virtually identical[12] with the semiclassical expression resulting from Eq. (39), the use of the semiclassical approximation in computing the first and third sums in Eq. (37) is justified.

5. EVALUATION OF THE CROSS SECTION: GENERAL FORM

We are now ready to evaluate the three summations given in Eq. (37).

It can be shown for the semiclassical case that the contribution to the *first summation* from a single collision in the time δ, in the limit of elastic scattering, becomes to high accuracy

$$\sum_s\{1\pm{}_s\eta_{c,c}(i)-{}_s\eta_{c,c}{}^2(2!)^{-1}-1\}P(s) \quad (49)$$

regardless of the size of $\alpha(n,k,s,n'-n)/(v)$ with respect to unity.[8] The plus sign is for E_f-E_e of Eq. (34) greater than zero; the minus sign is for the same expression less than zero. Using m as defined in Eq. (32) we have

$${}_s\eta_{c,c+m}=v^{-1}\{\alpha(n=(\epsilon_c+\hbar\omega_c)/\hbar\omega_c,k,s,m)-\alpha(n-1,k,s,m)\}. \quad (50)$$

In the limit of $n\gg1$, this may be approximated as

$${}_s\eta_{c,e}=(v)^{-1}(\partial\alpha/\partial n). \quad (51)$$

Finally,

$${}_s\eta_{c,c}{}^2=\sum_{m=m_1}^{m_2}{}_s\eta_{c,c+m}\cdot{}_s\eta_{c+m,c}. \quad (52)$$

[12] The difference in the results is traceable to the assumption of a constant velocity along the magnetic field in the semiclassical picture as opposed to the explicit introduction of v_i and v_f in the quantum approach.

The contributions from ${}_s\eta_{c,c}/(i\hbar)$ and ${}_s\eta_{c,c}\cdot{}_s\eta_{c,c}/[2(i\hbar)^2]$ are the adiabatic terms calculated in paper I. This is obvious since $m=0$ is the adiabatic condition of no change in the spatial part of the wave function represented by the quantum numbers n, k, and s. Hence the terms

$$\sum_{m_1,m\neq0}^{m_2}{}_s\eta_{c,c+m}\cdot{}_s\eta_{c+m,c}(2)^{-1} \quad (53)$$

come from nonadiabatic scattering.

After much algebra, expression (53) becomes[13]

$$-N_iv\delta2\pi\hbar(m\omega_c)^{-1}(2e^2/\hbar v)^2 2(4n)^{-1}\times\sum_{n'-n=1}^{m_2}(2)^{-1}\left(1+\frac{E_\beta}{E_z}\right)(n'-n)^2\{I_{n'-n}{}^2K_{n'-n-1}K_{n'-n+1}-K_{n'-n}{}^2I_{n'-n-1}I_{n'-n+1}\}. \quad (54)$$

For the *second summation* of Eq. (37) one has[14]

$$\sum_{n'-n=1}^{m_2}(n'-n)\hbar(2E_b)^{-1}\left\{(n'-n)(I,K)_{n',n}+nv_n\frac{d(v_n^{-1})}{dE_b}\bigg|_{E=\text{constant}}2(n'-n)\hbar\omega_c(I,K)_{n',n}+n\frac{d(I,K)_{n',n}}{dE_b}\bigg|_{E=\text{constant}}(n'-n)\hbar\omega_c\,,\right\} \quad (55)$$

where

$$d(v_n^{-1})/dE_b|_{E=\text{constant}}=(2v_nE_z)^{-1}. \quad (56)$$

For the *third summation* we have, again with the aid of Eq. (46), after performing the integration over the parameter s

$$-N_iv_n\delta\left(\frac{2\pi\hbar}{m\omega_c}\right)\left(\frac{2e^2}{\hbar v}\right)^2(2n)^{-1}\times\sum_{n'-n=1}^{m_2}(n'-n)^2(I,K)_{n',n}(2)^{-1}. \quad (57)$$

For all three summations there results, with the exclusion of adiabatic terms,

$$-N_iv_n\delta\left(\frac{2\pi\hbar}{m\omega_c}\right)\left(\frac{2e^2}{\hbar v_n}\right)^2(2n)^{-1}\times\sum_{n'-n=1}^{m_2}\left\{(2)^{-1}\left(1+\frac{E_b}{E_z}\right)(n'-n)^2(I,K)_{n',n}-\left(1+\frac{E_b}{E_z}\right)(n'-n)^2(I,K)_{n',n}-E_b(n'-n)^2\frac{d(I,K)_{n',n}}{dE_b}\bigg|_{E=\text{constant}}+\tfrac{1}{2}(n'-n)^2(I,K)_{n',n}\right\} \quad (58)$$

[13] See Appendix B.
[14] See Appendix C.

or

$$-N_i v_n\left(\frac{2\pi\hbar}{m\omega_c}\right)\delta\left(\frac{2e^2}{\hbar v_n}\right)^2(2n)^{-1}\sum_{n'-n=1}^{m_2}\left\{-(2)^{-1}\frac{E_b}{E_z}(I,K)_{n',n}\right.$$
$$\left.-E_b\frac{d(I,K)_{n',n}}{dE_b}\bigg|_{E=\text{constant}}\right\}(n'-n)^2. \quad (59)$$

The adiabatic terms add contributions[1]

$$\pm N_i i^{-1}(4\pi/3)(ec/H)\delta \quad (60)$$

and

$$N_i v_n(2\pi\hbar/m\omega_c)(2e^2/\hbar v_n)^2(2n)^{-1}(4)^{-1}. \quad (61)$$

The two terms correspond to

$$\pm i\int_{s=0}^{\infty} {}_s\eta_{c,c}\frac{\partial A}{\partial s}ds \quad \text{for} \quad E_f-E_e=\pm\hbar\omega_c \quad (62)$$

and

$$-\int_{s=0} {}_s\eta_{c,c}\cdot {}_s\eta_{c,c}2^{-1}\frac{\partial A}{\partial s}ds, \quad (63)$$

respectively, in Eq. (49), with $P(s)\equiv(\partial A/\partial s)$.

The contributions (60) and (62) correspond to the line *shift* calculated in Paper I. These contributions will be neglected in the following because it can be shown by detailed calculations that the numerical amount depends crucially on the manner of cutting off the wave functions at the maximum impact parameter, but that in general the resulting shift will be insignificant.

6. EVALUATION OF THE CROSS SECTION: STRONG COLLISION CUTOFF

In Eq. (37) finite limits m_1 and m_2 were introduced for the summation. The large values of m correspond to collisions for which the energy transfer between the directions perpendicular and parallel to the magnetic field is large compared to the cyclotron photon energy. Since our whole treatment is based on the Born approximation which is not valid for very strong collisions, a finite cutoff on the summation must be introduced. It will be shown that the contribution to the cross section from very strong collisions is identical to the ordinary Coulomb contribution without magnetic field in the straight-line approximation. This fact permits us to use the customary strong collision cutoff at the ninety degree deflection angle.

We investigate the summation in Eq. (59) by expanding $I_{n'-n}(x)$ and[15] $K_{n'-n}(x)$ for $(n'-n)\gg 1$. Writing

$$n'-n=p, \quad qn^{1/2}=x, \quad (64)$$

we have

$$I_p(x)=2^{1/2}(2\pi)^{-1}(p^2+x^2)^{-1/4}$$
$$\times\exp[(p^2+x^2)^{1/2}-p\sinh^{-1}(p/x)]$$
$$\times[\sum_{m=0}^{m=1}(-2)^m a_m\Gamma(m+\tfrac{1}{2})(p^2+x^2)^{-m/2}+O(x^{-m})] \quad (65)$$

and

$$K_p(x)=2^{-1/2}(p^2+x^2)^{-1/4}$$
$$\times\exp[-(p^2+x^2)^{1/2}+p\sinh^{-1}(p/x)]$$
$$\times[\sum_{m=0}^{m-1}(2)^m a_m\Gamma(m+\tfrac{1}{2})(p^2+x^2)^{-m/2}+O(x^{-M})], \quad (66)$$

with

$$a_0=1, \quad (67)$$

$$a_1=-(1/8)+(5/24)[1+(x^2/p^2)]^{-1}, \quad (68)$$

and

$$a_2=(3/128)-(77/576)(1+x^2/p^2)^{-1}$$
$$+(385/3456)(1+x^2/p^2)^{-2}. \quad (69)$$

To highest order in powers of p^{-1}, one obtains for $(I,K)_{n+p,n}$

$$I_p^2(x)K_{p-1}(x)K_{p+1}(x)-K_p^2(x)I_{p-1}(x)I_{p+1}(x)$$
$$=[2|p|^3(1+x^2/p^2)^{3/2}]^{-1}. \quad (70)$$

If the total change in E_z as the result of a collision, ΔE_z, is very much less than E_z, the contribution in terms of $n'-n$ can be expressed in terms of $\Delta\theta_z$, where $\theta_z=\cos^{-1}(E_b/E_z)^{+1/2}$. When in addition, ΔE_z is very much greater than $\hbar\omega_c$, the asymptotic form (70) can be used instead of the complete expressions (65) and (66).

Assuming ordinary Coulomb scattering and defining $\Delta\theta$ as the scattering angle away from the initial velocity of the particle, we have that the contribution to the scattering cross section in a region between $\Delta\theta$ and $\Delta\theta\approx d(\Delta\theta)$ is proportional to $(\Delta\theta)^{-1}d(\Delta\theta)$. The same contribution expressed in terms of $\Delta\theta_z$ varies as $(\Delta\theta_z)^{-1}d(\Delta\theta_z)$. However, the functional dependence of the contribution on $\Delta\theta$ and $\Delta\theta_z$ is the same as that obtained from the asymptotic form of $(I,K)_{n+p,n}$, Eq. (70). Hence the summation over $(I,K)_{n+p,n}$, which in the limit of large values of $n'-n$ may be replaced by an integration, can be converted into an integration over $\Delta\theta$ with $\pi/2$ as the upper limit.

This result is in agreement with the expectation that for very strong deflections the magnetic field should induce a negligible alteration on the behavior of the cross section. In the remainder of this section the conclusions summarized above will be verified and stated in a mathematically useful form.

Neglecting radiation, we derive a relationship between $n'-n$ and $\Delta\theta_z$ from the conservation of the total

[15] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2. p. 86.

electron energy during an electron-ion collision. Hence the electron kinetic energy is the same before and after a collision and

$$n\hbar\omega_c+\tfrac{1}{2}mv_z^2=n'\hbar\omega_c+\tfrac{1}{2}mv_z'^2 \qquad (71)$$

or

$$(n-n')\hbar\omega_c\approx mv_z\Delta v_z. \qquad (72)$$

Using the relationships

$$v_z=v\cos\theta_z, \quad v_b=v\sin\theta_z \qquad (73)$$

we obtain

$$\Delta v_z=-v\sin\theta_z\Delta\theta_z=-v_b\Delta\theta_z. \qquad (74)$$

Then with $v_b=(2E_b/m)^{1/2}$ we finally have

$$\Delta\theta_z=(n'-n)\hbar\omega_c/(2E_z^{1/2}E_b^{1/2}). \qquad (75)$$

After one converts the summation over $n'-n$ to a summation over $\Delta\theta_z$, it is obvious that the contribution to Eq. (59) from events between $\Delta\theta_z$ and $\Delta\theta_z+d(\Delta\theta_z)$ is proportional to $d(\Delta\theta_z)/\Delta\theta_z$ for $E_z\gg\hbar\omega_c$.

The next step consists in verifying the relationship between $\Delta\theta_z$ and $\Delta\theta$. We note that on the z axis $\theta_z=0$; on the other hand, $\theta=0$ is given by the direction of the initial electron velocity. We now consider a sphere in velocity space of radius v each point of which represents a possible electron velocity before or after the ion-electron collision. Next take the great circle through the points corresponding to the initial electron velocity and the intersection of the $\theta_z=0$ axis with the sphere. An angle can be defined between the planes of this great circle and the great circle obtained by joining the initial and final electron velocity points. Details are shown in Fig. 1.

We wish to express the coulomb contributions, proportional to $d(\Delta\theta)/\Delta\theta$, in terms of the alternate variables $d(\Delta\theta_z)$ and $\Delta\theta_z$. Consider the ring on the sphere such that $\Delta\theta\leq\Delta\theta_i\leq\Delta\theta+d(\Delta\theta)$. Since the contribution per unit length of the ring at angle α is independent of the value of α, the total contribution from the range of points i such that

$$\Delta\theta_z\leq(\Delta\theta_z)_i\leq\Delta\theta_z+d\Delta\theta_z \qquad (76)$$

and

$$\Delta\theta\leq\Delta\theta_i\leq\Delta\theta+d\Delta\theta \qquad (77)$$

is

$$4d(\Delta\theta_z)d(\Delta\theta)[2\pi(\Delta\theta)^2\sin\alpha]^{-1}. \qquad (78)$$

Here $d(\Delta\theta)/\Delta\theta$ is the contribution from the complete ring and $4d(\Delta\theta_z)/\sin\alpha$ is the total length of ring intercepted between $\Delta\theta_z$ and $\Delta\theta_z+d(\Delta\theta_z)$. Writing

$$\Delta\theta_z=\Delta\theta\cos\alpha \qquad (79)$$

we have for points such that $\Delta\theta_z$ is constant

$$d(\Delta\theta)/\Delta\theta=\tan\alpha d\alpha \qquad (80)$$

and the contribution (78) becomes

$$4d(\Delta\theta_z)d\alpha(2\pi\Delta\theta_z)^{-1}. \qquad (81)$$

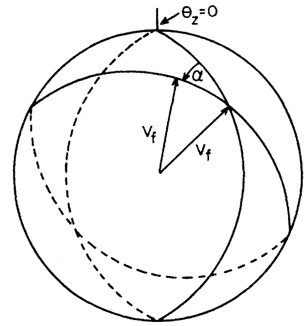

FIG. 1. The change in electron velocity due to a single elastic collision.

Since the range of integration on α is from 0 to $\pi/2$, the total contribution between $\Delta\theta_z$ and $\Delta\theta_z+d(\Delta\theta_z)$ is

$$d(\Delta\theta_z)(\Delta\theta_z)^{-1}. \qquad (82)$$

It is well known[16] that contributions to the collision cross section of the form $d(\Delta\theta)/\Delta\theta$ [or of the form $d(\Delta\theta_z)/\Delta\theta_z$] are obtained for the case of ordinary Coulomb scattering without a magnetic field. We therefore expect that the usage of our approximation, Eq. (70), whose contributions to the summation vary as $d(\Delta\theta_z)/\Delta\theta_z$, corresponds to the neglect of changes in the electron trajectories due to the presence of the magnetic field. It has thus been shown that the contribution to the cross section from very strong collisions is identical in form, for values between $\Delta\theta_z$ and $\Delta\theta_z+d(\Delta\theta_z)$, to the ordinary coulomb contribution without magnetic field in the straight-line approximation.

We are now ready to perform the summation over $n'-n$ with the appropriate cutoff corresponding to the angle $\Delta\theta=\pi/2$. We note that for the case of large magnetic quantum numbers n, there exist quantum numbers n^* such that $n\gg1$. The final result is insensitive to the numerical value of n^* as will be shown presently. Then we divide the terms of the summation into two groups according as to whether $n'-n>n^*$ *or* $n'-n<n^*$. For $n'-n\leq n^*$, with $\gamma=0.58$, Euler's constant, we have

$$\sum_{n'-n=1}^{n^*}\frac{1}{n'-n}=\ln(n^*)+\gamma. \qquad (83)$$

For $n'-n\geq n^*$ we have

$$\sum_a^b(n'-n)^{-1}\approx\int_a^b\frac{d(n'-n)}{(n'-n)}. \qquad (84)$$

Since

$$\Delta\theta_z(n'-n)\hbar\omega_c/(2E_b^{1/2}E_z^{1/2}), \qquad (85)$$

this may be written as

$$\int_{a\hbar\omega_c(2E_z^{1/2}E_z^{1/2})^{-1}}^{b\hbar\omega_c(2E_z^{1/2}E_b^{1/2})^{-1}}\frac{d(\Delta\theta_z)}{(\Delta\theta_z)}, \qquad (86)$$

[16] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 72.
[17] L. Oster, Astrophys. J. **137**, 332 (1963).
[18] E. Lifshitz, Zh. Eksperim. i Teor. Fiz. **7**, 390 (1937).

which may in turn be written as

$$2\int_{\Delta\theta_z=a\hbar\omega_c(2E_z^{1/2}E_b^{1/2})^{-1}}^{\Delta\theta_z=b\hbar\omega_c(2E_z^{1/2}E_b^{1/2})^{-1}} \frac{\beta d(\Delta\theta)}{(2\pi\Delta\theta)}, \qquad (87)$$

where β is the length in radians of the arc of radius $\Delta\theta$ lying within the area bounded by the limits of integration on the sphere of radius v; cf. Fig. 1.

Taking the cutoff angle for the collisions to $\Delta\theta=\pi/2$, we have for the summation over terms with $n'-n\geq n^*$

$$2\int_{\Delta\theta_z=(n^*\hbar\omega_c)(2E_z^{1/2}E_b^{1/2})^{-1}}^{\Delta\theta=\pi/2} \beta d(\Delta\theta)(2\pi\Delta\theta)^{-1}. \qquad (88)$$

Since

$$\beta=2[(\pi/2)-\sin^{-1}(n^*\hbar\omega_c/2E_z^{1/2}E_b^{1/2}\Delta\theta)], \qquad (89)$$

the integral becomes

$$\ln[\pi 2E_z^{1/2}E_b^{1/2}/(2n^*\hbar\omega_c)] - 2\hbar^{-1}\int_{x=\sin^{-1}(n^*\hbar\omega_c/(\pi E_z^{1/2}E_b^{1/2})}^{\pi/2} x\cot x dx, \qquad (90)$$

where $x=\sin^{-1}[n^*\hbar\omega_c/(2E_b^{1/2}E_z^{1/2}\Delta\theta)]$. The remaining integral can be shown to read $(\pi/2)\ln 2$ in the limit $n^*/n\ll 1$.

Therefore we finally obtain

$$\sum_{n'-n=1}^{\text{cutoff}} (n'-n)^{-1}=\ln[\pi E_z^{1/2}E_b^{1/2}/(2\hbar\omega_c)]+\gamma, \qquad (91)$$

so that

$$\sum_{n'-n=1}^{n_2} (I,K)_{n,n'}(n'-n)^2=E_z^{3/2}(2E^{3/2})^{-1} \times[\ln(\pi E_z^{1/2}E_b^{1/2}/2\hbar\omega_c)+\gamma]. \qquad (92)$$

7. EVALUATION OF THE CROSS SECTION: WEAK COLLISION CONTRIBUTIONS

The evaluation of the summation in the preceding section was based on approximation (70) for the Bessel functions which in effect neglected the magnetic field influence on the particle trajectories. This procedure is permissible for the larger values of $n'-n$. The first few terms however should be calculated more accurately.

For that purpose we include one more term in the expansion for the Bessel functions and obtain

$$\sum_{n'-n=1}^{m_2} (n'-n)^2(I,K)_{n',n} = \sum_{n'-n=1}^{m_2} (n'-n)^2\{2[x^2+(n'-n)^2]^{3/2}\}^{-1} \times\{1+A[x^2+(n'-n)^2]^{-1}+\cdots\}, \qquad (93)$$

where A is given by

$$a_0+a_1[1+x^2/(n^*-n)^2]^{-1}+a_2[1+x^2/(n'-n)^2]^{-2}. \qquad (94)$$

a_0, a_1, and a_2 are numerical constants whose values are immaterial for our purposes. Since

$$x=(n'-n)(E_b/E_z)^{1/2} \qquad (95)$$

as before, A is independent of the term number and one obtains for the above correction

$$\sum_{n'-n=1}^{m_2} (n'-n)^2A\{2[x^2+(n'-n)^2]^{5/2}\}^{-1} =0.6A/(1+E_b/E_z)^{5/2}. \qquad (96)$$

The value of the term for $n'-n=1$ alone is

$$0.5A/(1+E_b/E_z)^{5/2}. \qquad (97)$$

It is now apparent that the effect of deviations from coulomb trajectories is largely accounted for by using the first term of the series without any approximation and the remaining terms of the series in the coulomb approximation.

By this scheme of approximations the summation within Eq. (59) becomes

$$-(2)^{-1}\left(\frac{E_b}{E_z}\right)\{(I,K)_{n+1,n}+ +2^{-1}E_z^{3/2}E^{-3/2}[\ln(\pi E_z^{1/2}E_b^{1/2}/2\hbar\omega_c)+\gamma-1]\} -E_b\frac{d(I,K)_{n+1,n}}{dE_b}+3E_bE_z^{1/2}(4E^{3/2})^{-1} \times[\ln(\pi E_z^{1/2}E_b^{1/2}/2\hbar\omega_c)+\gamma-1]. \qquad (98)$$

In order to bring Eq. (98) into a more tractable form we consider the two limiting cases of particle motion predominantly perpendicular to or along the magnetic field lines.

Firstly, if $f=(E_b/E_z)^{1/2}\ll 1$, we have[8]

$$2^{-1}E_bE_z^{1/2}E^{-3/2}[\ln(\pi E_b^{1/2}E_z^{1/2}/2\hbar\omega_c)+\gamma]. \qquad (99)$$

On the other hand, if $f=(E_b/E_z)^{1/2}\gg 1$, Eq. (98) simplifies to

$$2^{-1}E_bE_z^{1/2}E^{-3/2}[\ln(\pi E_b/4\hbar\omega_c)+1.5]+\tfrac{1}{4}. \qquad (100)$$

For completeness, the quantity Q, defined by

$$Q=-E^{3/2}E_z^{-3/2}(I,K)_{n+1,n} -2E^{3/2}E_z^{-1/2}[d(I,K)_{n+1,n}/dE_b], \qquad (100a)$$

is illustrated in Fig. 2. This quantity, multiplied by the factor $2^{-1}E_bE_z^{1/2}E^{-3/2}$, is added to the term

$$2^{-1}E_bE_z^{1/2}E^{-3/2}[\ln(\pi E_z^{1/2}E_b^{1/2}/2\hbar\omega_c)+\gamma-1]$$

of Eq. (98) in the determination of the nonadiabatic contribution to the cross section.

8. PHYSICAL INTERPRETATION

The dimensionless quantities (99) and (100) which contain the significant portion of the cross section depend on the longitudinal and transverse energies in a manner which is not obvious at first glance. In order to gain a better understanding of the physical nature of this dependence we discuss in the following the behavior of the related quantity $(\Delta\theta_z)^2$ averaged over collisions. This value, $\langle(\Delta\theta_z)^2\rangle_{\text{av}}$, will be used to infer maximum impact parameters for nonadiabatic scattering which can be independently verified from a simple classical picture.

For $\langle(\Delta\theta_z)^2\rangle_{\text{av}}$ in time δ we have

$$\langle(\Delta\theta_z)^2\rangle_{\text{av}} = \sum_s \sum_{m=m_1}^{m_2} P(s) T_{m,c,s} m^2 (\hbar\omega_c/2E_b^{1/2}E_z^{1/2})^2 . \quad (101)$$

Using (37) and (57) one obtains

$$\sum_s \sum_{m=m_1}^{m_2} P(s) T_{m,c,s}(\delta) m^2 = N_i v_n \delta 2\pi\hbar (m\omega_c)^{-1} (2e^2/\hbar v_n)^2 (2n)^{-1} 8E_b^2 \times \sum_{n'-n=1}^{\infty} (n'-n)^2 2^{-1} (I,K)_{n,n'} . \quad (102)$$

Hence

$$(\Delta\theta_z)_{\text{av}}^2 = N_i v_n \delta 2\pi\hbar (m\omega_c)^{-1} (2e^2/\hbar v_n)^2 (2n)^{-1} 8E_b^2 \times (\hbar\omega_c/2E_b^{1/2}E_z^{1/2})^2 2^{-1} \sum_{n'+n=1}^{\infty} (n'-n)^2 (I,K)_{n,n'} . \quad (103)$$

Using (91) and (92), this is equal to

$$N_i v_n \delta 2\pi (m\omega_c)^{-1} (2e^2/\hbar v_n)^2 2E_b^2 (\hbar\omega_c/2E_b^{1/2}E_z^{1/2})^2 \times n^{-1} \{2^{-1} E_z^{3/2} E^{-3/2} [\ln(\pi E_z^{1/2} E_b^{1/2}/2\hbar\omega_c) + \gamma - 1] + (I,K)_{n+1,n}\} . \quad (104)$$

If we neglect insignificant numerical factors, the curly bracket becomes in the case of $(E_b/E_z)^{1/2} \gg 1$, using (100),

$$\text{const} \times \{2^{-1} E_z^{3/2} E^{-3/2} \ln(\pi E_z^{1/2} E_b^{1/2}/2\hbar\omega_c)\} . \quad (105)$$

On the other hand, for the case of $(E_b/E_z)^{1/2} \ll 1$, we have

$$\text{const} \times \{2^{-1} E_z^{3/2} E^{-3/2} \ln(\pi E_z/2\hbar\omega_c)\} . \quad (106)$$

If the scattering were of a pure Coulomb nature, $(\Delta\theta_z)_{\text{av}}^2$ would be proportional to $\ln[\pi/(2\Delta\theta_{\min})]$, where $(\Delta\theta_{\min})$ is a minimum scattering angle.

The values of $(\Delta\theta)_{\min}$ implied by this term are

$$(\Delta\theta)_{\min} = \hbar\omega_c/E_b^{1/2}E_z^{1/2} , \quad (E_b/E_z)^{1/2} \gg 1 , \quad (107a)$$

$$(\Delta\theta)_{\min} = \hbar\omega_c/E_z , \quad (E_b/E_z)^{1/2} \ll 1 . \quad (107b)$$

For the case of pure Coulomb scattering with the electron in a field of frequency ω_c the value of $(\Delta\theta)_{\min}$ is

$$(\Delta\theta)_{\min} = \hbar\omega_c/(E_b+E_z) . \quad (108)$$

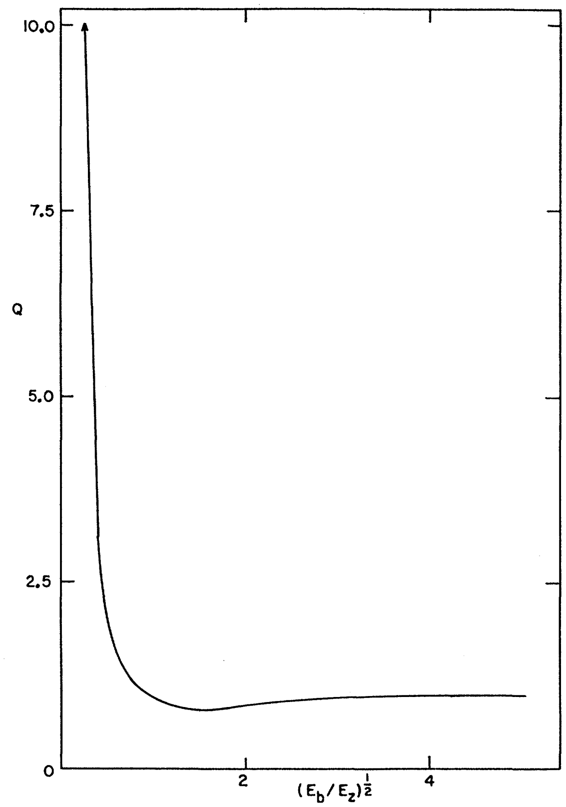

FIG. 2. Dependence of the quantity Q on the value $(E_b/E_z)^{1/2}$.

Since in the case $(E_b/E_z)^{1/2} \ll 1$ the particle motion is virtually a straight line, the values of $(\Delta\theta)_{\min}$ from Eqs. (107b) and (108) should agree closely. This can be verified upon inspection.

For $(E_b/E_z)^{1/2} \gg 1$, the value of $(\Delta\theta)_{\min}$ is larger by a factor of $(E_b/E_z)^{1/2}$ than the value in the pure coulomb case. To express (107a) in terms of an equivalent classical minimum scattering angle we assume

$$ze^2 v^{-1} = \hbar , \quad (109)$$

where v is the total particle velocity which is approximately equal to v_b for this case. Equation (109) is well justified in the pure coulomb case.[17]

$$(\Delta\theta)_{\min} = ze^2/[r_l(E_z/E_b)^{1/2}E] . \quad (110)$$

Equation (110) implies a maximum impact parameter of $r_l(E_z/E_b)^{1/2}$ to be used in the evaluation of contributions to the value of $\langle(\Delta\theta_z)^2\rangle$, where r_l is the electron Larmor radius.

This implication may be substantiated by calculating the increment in the electron velocity parallel to the magnetic field due to an ion-electron collision. We take for the electron coordinates

$$r = [r_l \cos(\omega_c t + \alpha) , r_l \sin(\omega_c t + \alpha) , v_z t] ,$$

and for the ion coordinates $R = (x,0,0)$. Then, as a first approximation, the force on the electron parallel to the magnetic field is[18]

$$-e^2(v_zt)k[x^2+r_l^2-2xr_l\cos(\omega_ct+\alpha)+(v_zt)^2]^{-3/2}. \quad (111)$$

The fractional change in velocity parallel to the magnetic field which determines the magnitude of $\Delta\theta_z$ is

$$v_z^{-1}\int_{-\infty}^{+\infty}\{-e^2v_ztm^{-1}[x^2+r_l^2-2xr_l\cos(\omega_ct+\alpha)+(v_zt)^2]^{-3/2}dt\}. \quad (112)$$

Expanding in increasing powers of $\cos(\omega_ct+\alpha)$, we find that the first nonvanishing term is of the form

$$-3e^2xr_l\int_{-\infty}^{+\infty}t(\sin\omega_ct)(\sin\alpha)dt[m(x^2+r_l^2+(v_zt)^2]^{-5/2}$$
$$=-3e^2m^{-1}xr_l\sin\alpha(\partial/\partial\omega_c)[2\omega_c^2v_z^{-3}\pi^{1/2}\Gamma(5/2)]^{-1}\times(x^2+r_l^2)^{-1}K_2[(\omega_c/v_z)(x^2+r_l^2)^{1/2}], \quad (113)$$

where we have used[19]

$$\int_0^\infty \cos x dx(x^2+z^2)^{-(n+\frac{1}{2})}=K_2(z)[\Gamma(5/2)]^{-1}\pi^{1/2}(2z)^{-2} \quad (114)$$

and

$$(\partial/\partial\omega_c)(\cos\omega_ct)=-t\sin\omega_ct. \quad (115)$$

Letting $x=\eta r_l$, where η is of the order of unity, one determines that the term (113) is equal to

$$-3e^2m^{-1}\eta\sin\alpha(\partial/\partial\omega_c)\times[\omega_c^2v_z^{-3}3^{-1}(1+\eta^2)^{-1}K_2(r_l\omega_cv_z^{-1}(1+\eta^2)^{1/2}]. \quad (116)$$

For $r_l\omega_c\gg v_z$, we have

$$K_2\alpha\exp\{-[r_l\omega_cv_z^{-1}](1+\eta^2)^{1/2}\}/[r_l\omega_cv_z^{-1}(1+\eta^2)^{1/2}]^{1/2}. \quad (117)$$

Therefore the contribution to the scattering angle is exponentially damped for $(E_b/E_z)^{1/2}\gg1$, if the ion is at a distance of the order of r_l from the electron's guiding center. This result is consistent with a shielding distance, $r_{\max}$, of order $r_l(E_z/E_b)^{1/2}$, for which, for an ion at a distance of the order of r_l from the electron, there would be no significant scattering effect for $(E_b/E_z)^{1/2}$ greater than unity.

It should be noted that for the case of $(E_b/E_z)^{1/2}\ll1$, too, one can obtain using Eq. (109) a value

$$r_{\max}=r_l(E_z/E_b)^{1/2} \quad (118)$$

which agrees with the value of $r_{\max}$ resulting from an expression analogous to Eq. (112).

Since we found that the scattering result in the classical picture is consistent with the classical extension of the quantum results, we can be confident that Eqs. (99) and (100) are derived on the correct physical grounds.

[19] A. Gray, B. G. Matthews, and T. M. MacRobert, *Bessel Functions* (Macmillan and Company, Ltd., London, 1922), p. 52.

9. LIMIT OF LOW MAGNETIC FIELD

In the case of weak magnetic field, i.e., for cases where the small angle scattering is limited by Debye shielding rather than by quantization in the magnetic field, it can be demonstrated that the half-width of the line which is proportional to the series expression (98) goes into the form derived in the absence of a magnetic field. In addition, it will be shown later[20] that under these conditions the scattering probabilities with and without magnetic eigenfunctions become identical. For the discussion of the half-width we note from Eq. (86) that the summation for states for which $n'-n\gg1$ can be made to correspond to an integration over the scattering angle $\Delta\theta$ for Coulomb scattering. Then the summation

$$\sum_{n'-n}(n'-n)^{-1}$$

is given by

$$\ln[(\pi/2)(\Delta\theta)_{\min}],$$

where $(\Delta\theta)_{\min}$ is the value of $\Delta\theta$ corresponding to an impact parameter at the shielding distance for the plasma. For a maximum impact parameter b, with E the particle energy,

$$(\Delta\theta)_{\min}=e^2(2bE)^{-1}, \qquad ze^2(\hbar v)^{-1}>1, \quad (119a)$$

$$(\Delta\theta)_{\min}=\hbar[b(2mE)^{1/2}]^{-1}, \quad ze^2(\hbar v)<1. \quad (119b)$$

The corresponding value for the series expression is then

$$2^{-1}E_bE_z^{1/2}E^{-3/2}\ln[(\pi/2)(\Delta\theta)_{\min}]. \quad (120)$$

This is the expected result.

SUMMARY OF RESULTS

For the real part of the expression γ in Eq. (25) we have:

(1) For the case of strong magnetic field, $\gamma_l=$ Larmor radius

(a) in general

$$\mathrm{Re}\gamma=-\nu_a=-2\pi\sqrt{2}n_ie^4m^{-1/2}\times[(2E^{3/2})^{-1}\langle\ln(\pi E_z^{1/2}E_b^{1/2}/2\hbar\omega_c)+0.42-E^{3/2}E_z^{-3/2}(I,K)_{n+1,n}-2E^{3/2}E_z^{-1/2}(d/dE_b)(I,K)_{n+1,n}\rangle+(4E_bE_z^{1/2})^{-1}]; \quad (121)$$

(b) for $(E_b/E_z)^{1/2}\ll1$,

$$-\nu_a=-2\sqrt{2}n_ie^4m^{-1/2}\{(2E^{3/2})^{-1}\times[\ln(\pi E_b/4\hbar\omega_c)+1.5]+(2E_bE_z^{1/2})^{-1}\}; \quad (122)$$

(c) for $(E_b/E_z)^{1/2}\gg1$,

$$-\nu_a=-2\pi\sqrt{2}n_ie^4m^{-1/2}\{(2E^{3/2})^{-1}\times[\ln(\pi E_b^{1/2}E_z^{1/2}/4\hbar\omega_c)+1.27]+(4E_bE_z^{1/2})^{-1}\}. \quad (123)$$

[20] See Appendix D.

(2) For the case of weak magnetic field, r_l=Larmor radius (b)

$$-\nu_a=-2\pi\sqrt{2}n_ie^4m^{-1/2}\times\{(2E^{3/2})^{-1}\ln[(\pi/2)(\Delta\theta)_{\min}]\}. \quad (124)$$

Note that in this case there is no adiabatic scattering contribution.

For $f_{b,a}$ as defined in Eq. (11) we obtain

$$f_{b,a}=\exp(-\nu_a\tau+i\Delta_{b,a}\tau)(b|\mu|a). \quad (125)$$

Now [cf. Eq. (9)]

$$I(\omega)\approx 2\text{ const}\times\omega^4\text{ Re}\sum_{\alpha,a}\rho_{\alpha,a}|\langle a|\mu|b\rangle|^2\times\{[\nu_a-i(\omega_c-\omega)]^{-1}+[\nu_a+i(\omega_c+\omega)]^{-1}\}$$
$$=2\text{ const}\times\omega^4\sum_{\alpha,a}\rho_{\alpha,a}|\langle a|\mu|b\rangle|^2\times\{\nu_a[\nu_a^2+(\omega_c-\omega)^2]^{-1}+\nu_a[\nu_a^2+(\omega_c+\omega)^2]^{-1}\}, \quad (126)$$

where to an accuracy of one part in n[21]

$$|\langle a|\mu_\alpha|b\rangle|=e|\langle a|x|b\rangle|=e|\langle a|y|b\rangle|=2^{-1}(n/\gamma)^{1/2}, \quad (127)$$

with

$$\gamma=eH/2c\hbar,\quad n=E_b/\hbar\omega_c \quad (128)$$

and

$$\sum_\alpha|\langle a|\mu_\alpha|b\rangle|^2=e^2E_bm\omega_c^2. \quad (129)$$

Defining $f(E_b,E_z^{1/2})$ as the probability of occupation of a quantum state with components E_b and $E_z^{1/2}$, we have

$$\rho_{\alpha,a}=n_lf(E_b,E_z^{1/2})dE_bdE_z^{1/2}\times\left[\int\int f(E_b,E_z^{1/2})dE_bdE_z^{1/2}\right]^{-1}, \quad (130)$$

and finally

$$I(\omega)=2\text{ const}\times\omega^4e^2(m\omega_c^2)^{-1}n_l\int\int E_bf(E_b,E_z^{1/2})\times\{\nu_a[\nu_a^2+(\omega_c-\omega)^2]^{-1}+\nu_a[\nu_a^2+(\omega_c+\omega)^2]^{-1}\}\times dE_bdE_z^{1/2}\bigg/\int\int f(E_b,E_z^{1/2})dE_bdE_z^{1/2}. \quad (131)$$

11. DETERMINATION OF LINE PROFILES

Using the results of the preceding section, it is interesting to obtain the emission profiles for several particle distributions. Since our results differ from earlier results most markedly in the angular dependence of the cross sections, one can obtain a picture of this effect by taking the two limiting cases of

(1) distributions with fixed total energy but isotropic in velocity space;

(2) distributions with $(E_b/E_z)^{1/2}\gg\ln(\pi E/4\hbar\omega_c)$ and fixed total energy. In both instances the velocity distributions change slowly compared to the times necessary for individual particle scattering, so that it is valid to neglect time variations in the distribution function.

For the isotropic case at fixed total energy, we have

$$I(\omega)\propto\sum_{+,-}\int_{\theta=0}^{\pi/2}\sin^3\theta\{\nu_a/[\nu_a^2+(\omega_c\pm\omega)^2]\}d\theta. \quad (132)$$

Here $\theta=0$ coincides with the magnetic field direction, and

$$(E_z/E)^{1/2}=\cos\theta,\quad (E_b/E)^{1/2}=\sin\theta. \quad (133)$$

If ν_a were independent of θ, the profile would be a simple Lorentzian shape. To obtain a measure of the deviation from Lorentzian, we take the angular-dependent part of the cross section to be the entire cross section when it is greater than the angular-independent part, and the angular-independent part to be the entire cross section when it is greater than the angular-dependent part.

Then using the results of Eqs. (122) and (123), and approximating the numerical factors 1.5 and 1.27 each by 1.4, we have on neglecting logarithmic contributions in the anisotropy:

$$\nu_a=2\pi\sqrt{2}m_ie^4m^{-1/2}(2E^{3/2})^{-1}E^{3/2}E_b^{-1}E_z^{-1/2};\quad 0\le E_b/E\le[\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}, \quad (134a)$$

$$\nu_a=2\pi\sqrt{2}m_ie^4m^{-1/2}(2E^{3/2})^{-1}[\ln(\pi E/4\hbar\omega_c)+1.4];\quad [\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}\le E_b/E,\quad [\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}\le 2E_z^{1/2}E^{-1/2}, \quad (134b)$$

$$\nu_a=2\pi\sqrt{2}n_ie^4m^{-1/2}(2E^{3/2})^{-1}E^{3/2}2^{-1}E_b^{-1}E_z^{-1/2};\quad 0\le 2E_z^{1/2}E^{-1/2}\le[\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}, \quad (134c)$$

where the term (134b) is obtained by taking the parts common to the two extreme anisotropic cases.

After approximating

$$\sin\theta=\cos[(\pi/2)-\theta]\approx\theta,\quad \theta\ll 1,$$

and using Eqs. (134a, b, c), we obtain from Eq. (132)

$$I(\omega)\propto\frac{\alpha}{(\omega-\omega_c)^2}\left[\ln\left(\frac{\pi E}{4\hbar\omega_c}\right)+1.4\right]^{-1}-\frac{2\alpha^2}{(\omega-\omega_c)^3}\tan^{-1}\left[\frac{[\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}}{2\alpha/(\omega-\omega_c)}\right]+\frac{2\alpha[\ln(\pi E/4\hbar\omega_c)+1.4]}{\{2\alpha[\ln(\pi E/4\hbar\omega_c)+1.4]\}^2+(\omega-\omega_c)^2}\times\{1-2^{-1}[\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}\}+\alpha 2^{-1}(\omega-\omega_c)^{-2}\ln\{1+(\omega-\omega_c)^2\alpha^{-2}2^{-1}\times[\ln(\pi E/4\hbar\omega_c)+1.4]^{-2}\}. \quad (135)$$

Here

$$\alpha=2\pi\sqrt{2}n_ie^4m^{-1/2}(2E^{3/2})^{-1}2^{-1}.$$

[21] Cf. Paper I, Sec. 2.

Relative to the Lorentzian $2\nu_a[(2\nu_a)^2+(\omega-\omega_c)^2]^{-1}$, where $\nu_a=2\alpha[\ln(\pi E/4\hbar\omega_c)+1.4]$, the intensity at $\omega-\omega_c=0$ is decreased by a fraction

$$\{4[\ln(\pi E/4\hbar\omega_c)+1.4]\}^{-1}, \tag{136}$$

while the "wings", for which

$$(\omega-\omega_c)\gg 2\alpha[\ln(\pi E/4\hbar\omega_c)+1.4],$$

are increased by a factor of

$$\frac{\ln((\omega-\omega_c)\{2\alpha[\ln(\pi E/4\hbar\omega_c)+1.4]\}^{-1})-1}{2[\ln(\pi E/4\hbar\omega_c)+1.4]}. \tag{137}$$

Since the line intensity at the Lorentzian half-width of $\{2\alpha[\ln(\pi E/4\hbar\omega_c)+1.4]\}$ is approximately

$$0.5\{1+0.1[\ln(\pi E/4\hbar\omega_c)+1.4]^{-1}\}$$

compared to a center value of unity, we see that the linewidth is slightly increased by the presence of the anisotropy.

A more refined treatment involving division of the angular part of ν_a into a constant part and another variable part always positive in sign, would have negligible effect on the correction in the wings, but it would cause an additional decrease in the center intensity and increase in the linewidth of the order of one part in $[\ln(\pi E/4\hbar\omega_c)+1.4]$.

Finally, for the distribution with $(E_b/E_z)^{1/2}\gg \ln(\pi E/4\hbar\omega_c)$, provided in addition that $(\omega-\omega_c)\ll \omega_c(E_z/E_b)^{1/2}$, the scattering produces no change in the particle distribution, and the emission with the use of Eq. (147), may be taken as

$$I(\omega)\propto \sum_{+,-} E_b\alpha[E^{3/2}/(E_bE_z^{1/2})] \times\{[\alpha E^{3/2}/(E_bE_z^{1/2})]^2+(\omega\pm\omega_c)^2\}^{-1}. \tag{138}$$

For the line center one then has

$$I(\omega=\pm\omega_c)\propto E_b^2E_z^{1/2}, \tag{139a}$$

while for the line wings there results

$$I(\omega)\propto \sum_{+,-} E_z^{-1/2}(\omega\pm\omega_c)^{-2}. \tag{139b}$$

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APPENDIX A. PHYSICAL MEANING OF THE QUANTUM NUMBER

In cylindrical coordinates, the Lagrangian L for a particle in a magnetic field[22] in the classical, nonrelativistic case is

$$L=2^{-1}m(\dot{r}^2+r^2\dot{\phi}^2+\dot{z}^2)-eHr^2\dot{\phi}(2C)^{-1}. \tag{A1}$$

For the generalized angular momentum p_ϕ we may write

$$p_\phi=\partial L/\partial\dot{\phi}=mr^2(\dot{\phi}-\omega_c/2). \tag{A2}$$

Imposing the quantum condition on p_ϕ leads to

$$p_\phi=\hbar l. \tag{A3}$$

Combining both expressions for p_ϕ we obtain

$$\hbar l+mr^2\omega_c/2=mr^2\dot{\phi}. \tag{A4}$$

Similarly, equating the classical and quantum expressions for the energy perpendicular to the magnetic field yields

$$n\hbar\omega_c=(m/2)(\dot{r}^2+r^2\dot{\phi}^2). \tag{A5}$$

Let b_0 be the distance of the particle's guiding center from the coordinate origin in the plane perpendicular to the magnetic field, and r_l denote the Larmor radius. Then

$$\begin{aligned}\dot{r}=0 \quad &\text{for} \quad r=b_0\pm r_l, \quad b_0>r_l,\\ \dot{r}=0 \quad &\text{for} \quad r=r_l\pm b_0, \quad r_l>b_0.\end{aligned} \tag{A6}$$

On squaring Eq. (A4) one obtains an equation for each value of r for which $\dot{r}=0$. Then substracting one of the resulting equations from the other we have

$$(m\omega_c2^{-1})^2(4r_lb_0)(2b_0^2+r_l^2)+\hbar lm\omega_c(4r_lb_0) =m^2(2n\hbar\omega_c)(4r_lb_0m^{-1}). \tag{A7}$$

Since $s=n-l$, this simplifies to

$$\pi b_0^2=2\pi\hbar s(m\omega_c)^{-1}. \tag{A8}$$

APPENDIX B. EVALUATION OF EXPRESSION (53)

To evaluate expression (53) we assume a random distribution of stationary ions with density N_i per cc, and take ΔA_s as the geometric area corresponding to an electron having collision variable s in the plane perpendicular to the magnetic field, relative to an ion fixed at the coordinate origin. The probability that the electron be in state s with respect to an ion in the time interval is $N_iv\delta\Delta A_s$, while the contribution to the first summation for a single collision with an electron in state s is

$$\sum_{m=1}^{m_2}[v^{-1}\frac{\partial\alpha}{\partial n}(n,k,s,m)]^2. \tag{B1}$$

The value of the summation (53) for time δ is then approximately

$$\sum_{m=1}^{m_2} N_iv\sum_{A_s}\left[\frac{\partial\alpha}{\partial n}(n,k,s,m)\right]^2 v^{-2}\Delta A_s\delta. \tag{B2}$$

Converting the summation over ΔA_s to an integral we have

$$N_iv\delta\sum_{n'-n=1}^{m_2}\int\frac{dA}{v^2}\left[\frac{\partial\alpha}{\partial n}(n,k,s,m)\right]^2. \tag{B3}$$

[22] A. Sokolov, Nuovo Cimento Suppl. 3, 743 (1956).

Since the distance b_0 between the projection of the guiding center of the electron orbit on the plane perpendicular to the magnetic field and the ion is related to s by the equation[8]

$$\pi b_0^2 = 2\pi\hbar s(m\omega_c)^{-1}, \tag{B4a}$$

we have

$$2\pi b_0 db_0 = 2\pi\hbar(m\omega_c)^{-1}ds. \tag{B4b}$$

The area which may be occupied by an electron guiding center located between b_0 and b_0+db_0 of an ion is $2\pi\hbar ds/(m\omega_c)$. Therefore we have for expression (B3)

$$N_i v\delta \int \frac{2\pi\hbar}{m\omega_c}\frac{ds}{v^2} \sum_{n'-n=1}^{m_2} \left[\frac{\partial\alpha}{\partial n}(n,k,s,m)\right]^2, \tag{B5}$$

or since s may vary from 0 to ∞,

$$N_i v \frac{2\pi\hbar}{m\omega_c}\left(\frac{2e^2}{i\hbar v}\right)^2 \delta \sum_{n'-n=1}^{m_2} K_{n'-n}'^2(qn^{1/2})$$

$$\times \int_{s=0}^{n} I_{n'-n}^2(qs^{1/2})\left(\frac{q}{2n^{1/2}}\right)^2 ds$$

$$+I_{n'-n}'^2(qn^{1/2})\int_{s=n}^{\infty} K_{n'-n}^2(qs^{1/2})\left(\frac{q}{2n^{1/2}}\right)^2 ds. \tag{B6}$$

Now

$$\int_a^b I_\beta^2(y)ydy = y^2(2)^{-1}[I_\beta^2 - I_{\beta-1}I_{\beta+1}]\Big|_a^b \tag{B7a}$$

and

$$\int_a^b K_\beta^2(y)ydy = -y^2(2)^{-1}[K_\beta^2 - K_{\beta-1}K_{\beta+1}]\Big|_a^b. \tag{B7b}$$

Then expression (B6) becomes

$$\left[N_i v\delta \frac{2\pi\hbar}{m\omega_c}\left(\frac{2e^2}{i\hbar v}\right)^2 \frac{2}{4n}\right]$$

$$\times \sum_{n'-n=1}^{m} K_{n'-n}'^2[qn^{1/2}]\left\{\frac{(qn^{1/2})^2}{2}[I_{n'-n}^2(qn^{1/2})\right.$$

$$\left.-I_{n'-n-1}(qn^{1/2})I_{n'-n+1}(qn^{1/2})]\right\}+I_{n'-n}'^2[qn^{1(2}]$$

$$\times\left\{\frac{(qn^{1/2})^2}{2}[K_{n'-n+1}(qn^{1/2})K_{n'-n-1}(qn^{1/2})\right.$$

$$\left.-K_{n'n}^2(qn^{1/2})]\right\}. \tag{B8}$$

Using the relations

$$I_{a-1}(z)+I_{a+1}(z) = 2I_a'(z), \tag{B9a}$$

$$K_{a-1}(z)+K_{a+1}(z) = -2K_a'(z) \tag{B9b}$$

to eliminate the derivatives, we obtain for the sum in (B8)

$$\sum_{n'-n=1}^{m} \tfrac{1}{4}[K_{n'-n-1}(qn^{1/2})+K_{n'-n+1}]^2$$

$$\times\{(qn^{1/2})^2(2)^{-1}[I_{n'-n}^2 - I_{n'-n-1}I_{n'-n+1}]\}$$

$$+\tfrac{1}{4}(I_{n'-n-1}+I_{n'-n+1})^2$$

$$\times\{(qn^{1/2})^2(2)^{-1}[K_{n'-n-1}K_{n'-n+1}-K_{n'-n}^2]\}. \tag{B10}$$

Furthermore,

$$I_{a-1}(z)-I_{a+1}(z) = 2a(z)^{-1}I_a(z), \tag{B11a}$$

$$K_{a-1}(z)-K_{a+1}(z) = -2a(z)^{-1}K_a(z). \tag{B11b}$$

Hence instead of (B10) we have after some algebra

$$\sum_{n'-n=1}^{m_2} \frac{1}{4}\left[\frac{4(n'-n)^2}{(qn^{1/2})^2}K_{n'-n}^2 + 4K_{n'-n-1}'K_{n'-n+1}\right]$$

$$\times\{(qn^{1/2})^2(2)^{-1}[I_{n'-n}^2 - I_{n'-n-1}I_{n'-n+1}]\}$$

$$+\tfrac{1}{4}[4(n'-n)^2(qn^{1/2})^{-2}I_{n'-n}^2 + 4I_{n'-n-1}I_{n'-n+1}]$$

$$\times\{(qn^{1/2})^2(2)^{-1}[K_{n'-n}^2 - K_{n'-n-1}K_{n'-n+1}]\} \tag{B12}$$

and, upon replacing q according to Eq. (40),

$$\sum_{n'-n=1}^{m} \tfrac{1}{2}(n'-n)^2[1+(E_b/E_z)^{-1}]$$

$$\times[I_{n'-n}^2 K_{n'-n-1}K_{n'-n+1}$$

$$-K_{n'-n}^2 I_{n'-n-1}I_{n'-n+1}]. \tag{B13}$$

Expression (54) then results directly.

APPENDIX C. SECOND SUMMATION IN EQ. (37)

For the second summation in Eq. (37) we note that the contribution from the Born approximation to $T_{m,c,s}(\delta)$ for a single collision in time δ is

$$(v_c/v_e)\left(c,s\left|\int_{-\infty}^{\infty} U_0^{-1}H_v(t)U_0 dt\right|e\right)$$

$$\times\left(f\left|\int_{-\infty}^{\infty} U_0^{-1}H_v(t)U_0 dt\right|d\right). \tag{C1}$$

Assuming a random location of ions we have for the number of such collisions for a single electron in the time

$$N_i v_c \delta\Delta A_s. \tag{C2}$$

Combining (C1) and (C2), one obtains

$$T_{m,c,s}(\delta) = N_i v_c \delta\Delta A_s \frac{v_c}{v_e}\left[c,s\left|\int_{-\infty}^{\infty} U_0^{-1}H_v(t)U_0 dt\right|e\right]$$

$$\times\left[f\left|\int_{-\infty}^{\infty} U_0^{-1}H_v(t)U_0 dt\right|d\right]. \tag{C3}$$

Since

$$\epsilon_f = \epsilon_e \pm \hbar\omega_c, \quad \text{(C4)}$$

the difference between the contributions for $\epsilon_e = \epsilon_c + m\hbar\omega_c$ and those for $\epsilon_e = \epsilon_c - m\hbar\omega_c$ is altered negligibly by replacing e for f, and c for d in the second matrix element of Eq. (C1). One then has in place of (C1)

$$(v_c/v_e)\left[c,s\left|\int_{-\infty}^{\infty} U_0^{-1}H_\nu(t)U_0 dt\right|e\right] \times\left[e\left|\int_{-\infty}^{\infty} U_0^{-1}H_\nu(t)U_0 dt\right|c\right]. \quad \text{(C5)}$$

In Sec. (4) this last expression was shown to be the semiclassical analog of the quantum mechanical expression (w/v_i) in Eq. (46). Therefore we may write in terms of w

$$T_{m,c,s}(\delta) = N_i v_c \delta \Delta A_s (w/v_c). \quad \text{(C6)}$$

Here ΔA_s is, as before, the geometric area corresponding to an electron having quantum state s with respect to a fixed ion origin, N_i is the ion density, and $v_c(=v_i)$ is the initial electron velocity parallel to the magnetic field.

For the second summation we therefore have

$$N_i v_n \delta\left(\frac{2\pi\hbar}{m\omega_c}\right)\left(\frac{2e^2}{\hbar v_n}\right)^2 \sum_{n'-n=m_1}^{m_2} n_e E_b^{-1/2} \times[I_{n'-n}{}^2 K_{n'-n-1}K_{n'-n+1} - K_{n'-n}{}^2 I_{n'-n-1}I_{n'-n+1}] \times(2E_b)^{-1}(n'-n)\hbar\omega_c(v_n/v_{n'}), \quad \text{(C7)}$$

where n_e is the lesser of n and n', and

$$E_b = n_e\hbar\omega_c, \quad E_z = \frac{1}{2}\left(\frac{m}{2}v_n{}^2 + \frac{m}{2}v_{n'}{}^2\right). \quad \text{(C8)}$$

Defining

$$(I,K)_{n',n} \equiv [I_{n'-n}{}^2 K_{n'-n-1}K_{n'-n+1} - K_{n'-n}{}^2 I_{n'-n-1}I_{n'-n+1}] \quad \text{(C9)}$$

and pairing off contributions from $\pm|n'-n|$, we obtain the result of Eq. (55).

APPENDIX D. SIMILARITY OF SCATTERING PROBABILITIES WITH AND WITHOUT MAGNETIC FUNCTIONS

Using Eqs. (37) and (57), the probability, $\omega(\Delta n)$, for the incident electron beam of one particle/cm² sec to be scattered from the state with energy $E_b' = E_b \pm \Delta n\hbar\omega_c$ as the result of encounters with a single ion is

$$\omega(\Delta n) \approx \left(\frac{2e^2}{\hbar v}\right)^2\left(\frac{2\pi\hbar}{m\omega_c}\right) n[I_{n'-n}{}^2(qn^{1/2})K_{n'-n+1}K_{n'-n-1} - K_{n'-n}{}^2 I_{n'-n+1}I_{n'-n-1}]. \quad \text{(D1)}$$

Equation (D1) holds in the Born approximation, a necessary condition for which is $\hbar\omega_c \ll \Delta E_b \ll E_b$. Asymptotically, using the result of Eq. (70) this becomes

$$\omega(\Delta n) \approx \left(\frac{2e^2}{\hbar v}\right)^2\left(\frac{2\pi\hbar}{m\omega_c}\right) n[2(\Delta n)^{-3}]^{-1} E_z^{3/2} E^{-3/2}. \quad \text{(D2)}$$

Taking

$$\theta_z = \cos^{-1}(v_z/v) \quad \text{(D3)}$$

and defining $\omega(\Delta\theta_z)d(\Delta\theta_z)$ as the probability for an electron to be scattered with angle $\Delta\theta_z$ and $\Delta\theta_z + d(\Delta\theta_z)$, we have

$$\omega(\Delta\theta_z)d(\Delta\theta_z) = \omega(\Delta n)\left[\frac{d}{d(\Delta E_b)}\Delta n\right]\left[\frac{d(\Delta E_b)}{d(\Delta\theta_z)}\right] d(\Delta\theta_z). \quad \text{(D4)}$$

Using

$$\Delta E_b = \hbar\omega_c \Delta n = \Delta(m v_z{}^2)/2 = m v_z \Delta v_z = m v_z v_b \Delta\theta_z \quad \text{(D5)}$$

as well as

$$d(\Delta E_b) = m v_z v_b d(\Delta\theta_z), \quad \text{(D6)}$$

we obtain as the result from magnetic eigenfunctions

$$\omega(\Delta\theta_z)d(\Delta\theta_z) = 2\pi e^2 4^{-1} E_z^{-1/2} E^{-3/2} d(\Delta\theta_z)(\Delta\theta_z)^{-3}. \quad \text{(D7)}$$

We now compare this result with the case of Coulomb scattering into angles $\Delta\theta$ and ϕ (azimuth). The scattering probability for a beam with one particle/cm²-sec into the solid angle between $\Delta\theta$ and $\Delta\theta + d(\Delta\theta)$ and ϕ and $\phi + d\phi$ on the sphere of speed $v = (2E/m)^{1/2}$ in velocity space reads

$$W(\Delta\theta,\phi)d\Omega = \omega(\Delta\theta,\phi)\sin(\Delta\theta)d(\Delta\theta)d\phi, \quad \text{(D8)}$$

with

$$\omega(\Delta\theta,\phi) = 4^{-1}(e^2/2E)^2[\sin(\Delta\theta/2)]^{-4}. \quad \text{(D9)}$$

For small $\Delta\theta$,

$$\sin\Delta\theta \approx \Delta\theta \quad \text{(D10)}$$

and the probability becomes

$$e^4 E^{-2} d(\Delta\theta)d\phi(\Delta\theta)^{-3}. \quad \text{(D11)}$$

We define the aximuth angle so that $\phi = 0$ coincides with a line of constant latitude in the sphere of speed v in velocity space. Then

$$\Delta\theta = \sin\phi(\Delta\theta), \quad \text{(D12)}$$

and for constant ϕ,

$$d(\Delta\theta_z) = \sin\phi d(\Delta\theta). \quad \text{(D13)}$$

Equation (D8) becomes

$$e^4 E^{-2} d(\Delta\theta_z)\sin^3\phi d\phi(\sin\phi)^{-1}(\Delta\theta_z)^{-3}. \quad \text{(D14)}$$

For the zone on the sphere of speed v in velocity space with values of $\Delta\theta_z$ between $\Delta\theta_z$ and $\Delta\theta_z + d(\Delta\theta_z)$, ϕ

varies between $-\pi/2$ and $\pi/2$. Taking these values as limits of integration on ϕ, we obtain for the scattering probability into the zone $d(\Delta\theta_z)$

$$\omega(\Delta\theta_z)d(\Delta\theta_z)=(\pi/2)e^4E^{-2}d(\Delta\theta_z)(\Delta\theta_z)^{-3}. \quad \text{(D15)}$$

This is the pure coulomb result.

The scattering probability in the magnetic case tacitly assumed one particle/cm^2-sec in the direction of the magnetic field incident on the scatterer. The scattering probability in the coulomb case assumed one particle/cm^2-sec in the direction of the velocity incident on the scatterer. Since the assumption in the coulomb description is the equivalent of $E_z^{1/2}/E^{1/2}$ particles/cm^2-sec in the direction of the magnetic field, the value for this situation should be a factor of $E_z^{1/2}/E^{1/2}$ lower than the magnetic value.

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Spin Diffusion in Gases at Low Temperatures*†

V. J. Emery
Department of Physics, University of California Berkeley, California
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It is shown that the first approximation to the spin-diffusion coefficient D of a gas at low temperatures involves a scattering cross section for distinguishable particles only, so that D is different from the self-diffusion coefficient D_0. Quantum symmetry effects show up in the second approximation to D but the correction to the first approximation is small. The theoretical values of D for gaseous hydrogen and gaseous He^3 agree quite well with experimental results.

I. INTRODUCTION

RECENTLY, nuclear magnetic resonance experiments have been used to measure the spin-diffusion coefficient D in gaseous hydrogen[1,2] between 20 and 55°K, and[3] in gaseous He^3 between 1.7 and 4.2°K. In these experiments, it is usually assumed that D is identical with the self-diffusion coefficient[4] D_0 of the gas and that the nuclear spin is merely a label which allows the diffusion to be observed. However, it turns out that the values[5,6] of D_0 given by the Chapman-Enskog theory of transport processes are systematically smaller than the experimental values of D, and that they lie outside the limits of experimental error.

The object of this paper is to show that, in fact, D_0 is not the quantity measured in these experiments and that an appropriate expression for D reproduces the experimental results quite well. The distinction between D_0 and D arises only in those situations in which it is necessary to treat the scattering of particles quantum mechanically. For a two-component gas at the temperatures under consideration, the coefficient of diffusion of component 1 relative to component 2 is given by[4]

$$D_{12}=\frac{3}{8nm}\frac{kT}{\Omega_{12}^{(1,1)}}, \quad (1)$$

where T is the temperature, k is Boltzmann's constant, n the total number density, and m the mass of the particles (assumed to be the same for each component). Quantum mechanical effects enter through $\Omega_{12}^{(1,1)}$ which is a special case of

$$\Omega_{12}^{(n,t)}=\left(\frac{kT}{\pi m}\right)^{1/2}\int_0^\infty d\gamma e^{-\gamma^2}\gamma^{2t+3}Q_{12}^{(n)}(\gamma), \quad (2)$$

where γ^2 is the relative kinetic energy of the pair of particles divided by kT, and

$$Q_{12}^{(n)}=\frac{\pi}{\gamma}\left(\frac{m}{kT}\right)^{1/2}\int_0^\pi dx \sin x(1-\cos^n x)\alpha_{12}(\gamma,x). \quad (3)$$

$\alpha_{12}(\gamma,x)$ is proportional to the differential cross section for scattering of a particle from component 1 by a particle from component 2 at a relative kinetic energy γ^2kT. x is the scattering angle.

The self-diffusion coefficient D_0 is defined as the limit of D_{12} when components 1 and 2 become identical, and $\alpha_{12}(\gamma,x)$ is taken to be proportional to the properly symmetrized differential cross section for the scattering of identical particles. Thus, for example, in pure orthohydrogen, the particles have nuclear spin $I=1$ and rotational angular momentum $J=1$, and D_0 has

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